Mathematics bilingual

This course was one of the two mathematical courses and our topic was an introduction to basic concepts of elementary differential geometry, illustrated by their application to the Newtonian theory of gravitation in three space dimensions. Elementary differential geometry refers to the mathematical theory describing and analyzing geometric properties of curves in Euclidean space.

As a first part of the course we addressed vector algebra and arithmetics, choosing an abstract approach to vector spaces and considering the standard Euclidean spaces as illustrating examples. After a short introduction to basic physical concepts like velocity, acceleration, force and momentum, the second part of the course was on differential calculus, motivated by physical problems and basically following the historical development of Newtonian mechanics and differential calculus. This included the conduction of small physical experiments which helped obtaining physical and geometric intuition for the mathematical concepts involved.

Apart from the scientific contents, another goal of our course was to use and teach English as main medium of communication. We reached that goal to an surprising extent and are very impressed how easily our participants could cope with that additional challenge. As we taught the whole course in English language, our documentation is in English, too.

All in all we are very content with the results of our course and had a nice and enjoyable time with all the participants during the Science Academy.

Rainer Mühlhoff and Momsen Reincke

The participants

We were the Mathematic-Bilingual Course! They called us "bilingual", although our course actually was only in English language. But with Momsen and Rainer, our smart course leaders, we always thought that the English language just gave the icing on the cake to our perfect course.

Rainer Mühlhoff (Kursleiter)

The heart of our course. He moved us by multiknowledge and gentle irony.

Momsen Reincke (Kursleiter)

Rainer would have been lost in space without him. His courses got funny because of his handwriting.

Anastasia Dietrich

The "dimension-girl".

Bo Song

We won't forget his presentations...

Bendix Labeit

Abstraction-king of mathematical groups.

Tan Lou

"Louise-multilingual" Chinese, German and English!

Lauretta Schwarz

Absolutely (linearly) independent.

Magdalena Wandelt

She always arranged for confusing questions.

Annika Konzelmann

She made the sequence converging.

Heike Hummel

Our trigonometry pro.

Mario Schulz

The quantum and nuclear fusion expert.

Christoph Sünderhauf

Walking proof that masterminds are mostly tiny.

David Kranzhöfer

Likes throwing vectors.

Jonathan Schaible

Joker to solve every problem.

Marian Huß

He perfectly did "poetic mathematics".

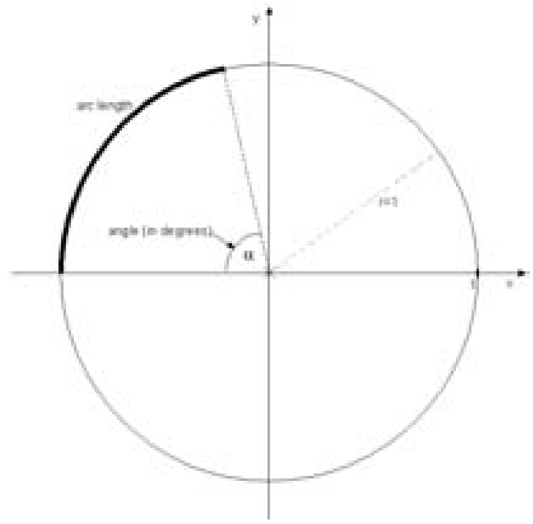
Jan Forstbauer

Our star during the "Highland-Games".

Trigonometric Functions

Heike Hummel, David Kranzhöfer, Jonathan Schaible, Christoph Sünderhauf

You can measure angles either in arc length or in degrees.



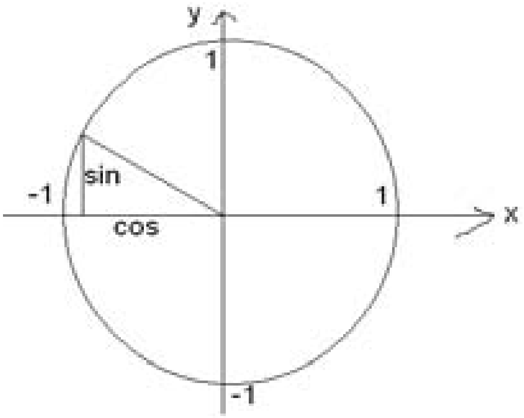
In the unit circle with radius $r = 1$ the arc length of a full turn equals 2π. Therefore, the arc length of an angle can be calculated

$$\frac{\alpha}{360°} \cdot 2\pi = \frac{\alpha}{180°}\pi\,.$$

Here are some example conversions:

degrees	arc length
0°	0
45°	$\frac{1}{4}\pi$
90°	$\frac{1}{2}\pi$
180°	1π
360°	2π

If we now draw this unit circle into a coordinate system and mark at an arbitrary angle α the point on the circle we get 2 coordinates. The first coordinate gives us a point on the x-axis, which is the so called cosine of α ($\cos(\alpha)$), the second point gives us a point on the y-axis, which is the so called sine of α ($\sin(\alpha)$).

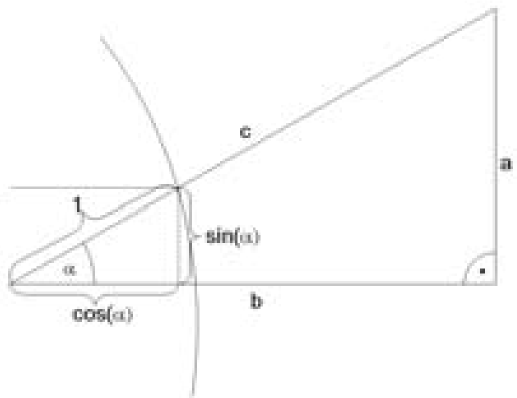


Sine and cosine at the Unitcircle

Sine and cosine are defined for every angle α. Here are some examples:

α [arc]	$\cos(\alpha)$	$\sin(\alpha)$
0	1	0
$\frac{1}{4}\pi$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
π	-1	0
$\frac{1}{2}\pi$	0	1
$\frac{3}{2}\pi$	0	-1

Let's consider a triangle with the side lengths a, b, c and the angle α.



What can we now say about the proportions of the side lengths if we know the angle? We know from the so called Strahlensatz, that

$$\frac{\sin(\alpha)}{a} = \frac{1}{c} \quad \Leftrightarrow \quad \sin(\alpha) = \frac{a}{c}\,.$$

Now we can do the same with $\cos(\alpha)$:

$$\frac{\cos(\alpha)}{b} = \frac{1}{c} \quad \Leftrightarrow \quad \cos(\alpha) = \frac{b}{c}\,.$$

The Pythagorean theorem together with the above result yields:

$$c^2 = a^2 + b^2$$

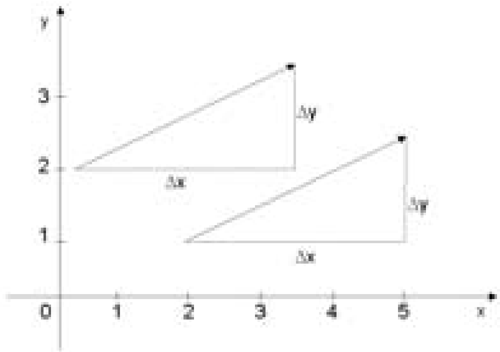
$$\Leftrightarrow 1 = \frac{a^2+b^2}{c^2} = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \sin^2(\alpha) + \cos^2(\alpha)\,.$$

Vectors as Arrows

Bendix Labeit

To start our discussion of vectors, consider classes of arrows in the two-dimensional plane all having the same length and direction. Each class can be described by two real numbers, the x and the y displacement. Therefore each element of $\mathbb{R}^2 := \{(x, y) \,|\, x, y \in \mathbb{R}\}$ can be identified with a class of arrows all having the same length and direction and is called a vector. We denote an element $v \in \mathbb{R}^2$ as

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}.$$



Both arrows represent the same vector.

There are also other vectors and we will give a precise and general definition of the concept "vector" later.

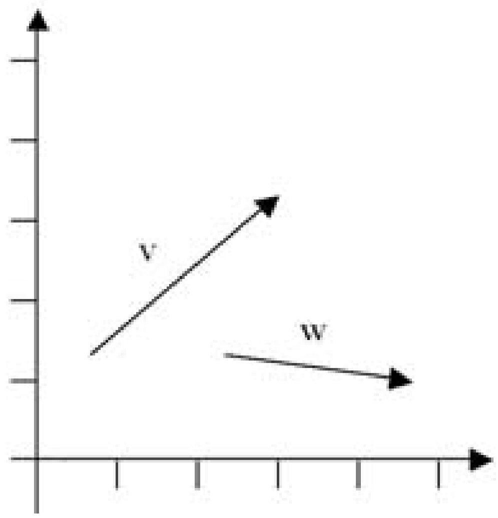
Vector arithmetics

Marian Huss

As you already know, vectors are classes of arrows all having the same length and the same direction. When we write down the components of a vector, those are always real numbers. So it is straight forward to introduce simple rules for vector calculation based on calculations with real numbers. When we calculate with vectors we call this *vector arithmetics*.

Vector addition

Adding up two vectors is not very difficult. The only thing we have to do is to "connect" the two corresponding arrows somehow. "Connecting two vectors" fulfills, as we will see in the following paragraph, all axioms of a commutative group of a real vector space (Hint: If you haven't done already, please read the section about "commutative groups" and "vector spaces" first!). So now we'll give a geometrical definition of the addition of two vectors.

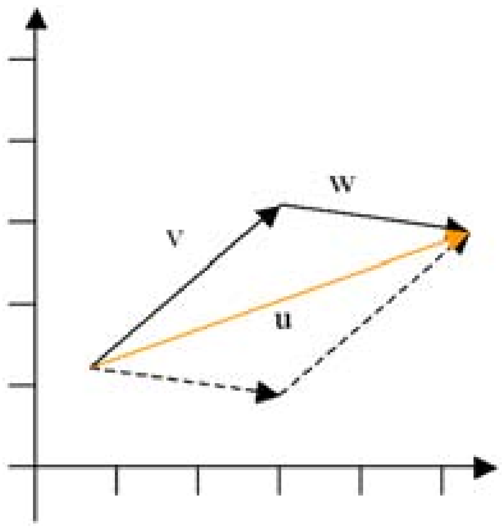


This picture shows two vectors $\vec{v}$ and $\vec{w}$ in the 2-dimensional Euclidean plane.

$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} c \\ d \end{pmatrix}$$

To add these two vectors, we move the tail (starting point) of $\vec{w}$ to the tip (ending point) of $\vec{v}$ in order to connect them (we are allowed to do this because the components of the vectors do not change).

This method is shown in the following picture:



We connect the tail of $\vec{v}$ with the tip of $\vec{w}$ and get as a result another vector $\vec{u}$.

The components of that third vector $\vec{u}$ are exactly the sums of the x-components and the y-components of $\vec{v}$ and $\vec{w}$ respectively:

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$$

$$\vec{u} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$$

Example 1.

$$\vec{v} = \begin{pmatrix} 2 \\ 1,5 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 3,5 \\ -0,5 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1,5 \end{pmatrix} + \begin{pmatrix} 3,5 \\ -0,5 \end{pmatrix} = \begin{pmatrix} 2+3,5 \\ 1,5-0,5 \end{pmatrix} = \begin{pmatrix} 5,5 \\ 1 \end{pmatrix}$$

$$\vec{u} = \begin{pmatrix} 5,5 \\ 1 \end{pmatrix}$$

We can also prove the **addition law of commutativity** for vector addition geometrically:

We just take $\vec{w}$ as the first vector and take as its tail exactly the point we took as the tail of $\vec{v}$ before. Then we put the tail of $\vec{v}$ at the tip of $\vec{w}$ and see that the tip of $\vec{v}$ is exactly the point that has been the tip of $\vec{w}$ after the first addition. Using formulas this means:

The $\vec{u}$ we get as a result of that second addition is exactly the first vector $\vec{u}$.

$$\begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c+a \\ d+b \end{pmatrix}$$

$$\vec{u} = \begin{pmatrix} c+a \\ d+b \end{pmatrix}$$

So now, as a final result, we have got the formula of vector addition:

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$$

Of course this rule holds for additions of more than two vectors, too.

We now give a proof of the **law of associativity** for vector addition, which says:

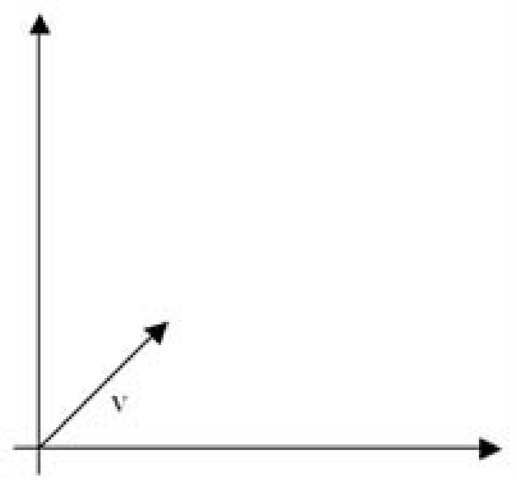
$$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}\,.$$

Everyone should be able to understand the following proof now:

$$\begin{aligned} \vec{u} + (\vec{v} + \vec{w}) &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix} \\ &= \begin{pmatrix} u_1 + v_1 + w_1 \\ u_2 + v_2 + w_2 \end{pmatrix} \\ &= \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= (\vec{u} + \vec{v}) + \vec{w} \end{aligned}$$

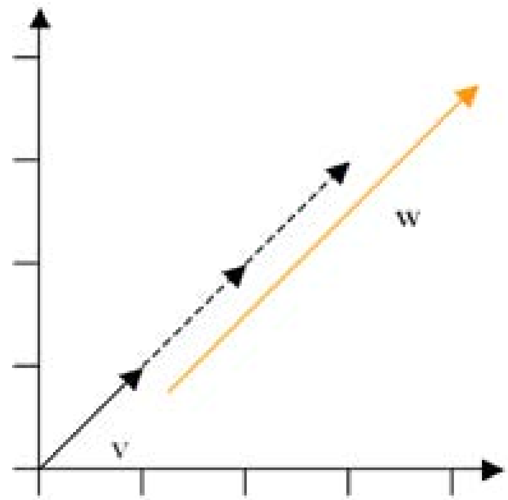
Scalar multiplication

Addition is not the only kind of vector arithmetics. It is also possible to "*scale*" a vector. "Scaling a vector" always means to multiply it by a so called "*scalar*". This scalar is just any real number. When we mention a scalar in a general mathematical formula, we usually use the Greek letter λ to symbolize it.



The picture above shows a vector $\vec{v}$ in the Euclidean plane with

$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}.$$



In this picture we see another vector $\vec{w}$. It has got the same direction as $\vec{v}$ and its length is three times the length of $\vec{v}$.

As you can see, $\vec{w}$ has got the components

$$\begin{pmatrix} 3 \cdot a \\ 3 \cdot b \end{pmatrix} = \begin{pmatrix} 3 \cdot v_1 \\ 3 \cdot v_2 \end{pmatrix}.$$

Now that you already know what $\vec{v}$ and $\vec{w}$ have to do with each other (same direction; $\vec{w}$ has got 3 times the length of $\vec{v}$), you may easily see that it makes sense to write

$$3 \cdot \vec{v} = \vec{w}$$

because

$$3 \cdot \vec{v} = 3 \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3 \cdot v_1 \\ 3 \cdot v_2 \end{pmatrix} = \vec{w}\,.$$

So in this case, 3 is the scalar of the multiplication and we finally know the general formula of scalar multiplication:
For all $\lambda \in \mathbb{R}$ holds:

$$\lambda \cdot \vec{v} = \lambda \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda \cdot v_1 \\ \lambda \cdot v_2 \end{pmatrix}$$

Example 2.

$$\vec{v} = \begin{pmatrix} 1,5 \\ 1,5 \end{pmatrix}$$

$$\begin{aligned} 3 \cdot \vec{v} = 3 \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= 3 \cdot \begin{pmatrix} 1,5 \\ 1,5 \end{pmatrix} \\ = \begin{pmatrix} 3 \cdot 1,5 \\ 3 \cdot 1,5 \end{pmatrix} &= \begin{pmatrix} 4,5 \\ 4,5 \end{pmatrix} \end{aligned}$$

At last, in order to repeat both kinds of vector arithmetics which we've learned something about now, we want to present the **proof of distributivity** for vector addition. For that we need to know the scalar multiplication.
The law of distributivity says:

$$\lambda \cdot (\vec{v} + \vec{w}) = \lambda \cdot \vec{v} + \lambda \cdot \vec{w}$$

Proof:

$$\begin{aligned} \lambda \cdot \vec{v} + \lambda \cdot \vec{w} &= \lambda \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \lambda \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= \lambda \cdot \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix} \\ &= \lambda \cdot (\vec{v} + \vec{w}) \end{aligned}$$

Vector Spaces

Magdalena Wandelt

After we talked about vectors, we came to the topic "vector spaces". The set of vector-space-axioms is something like a "wishlist", and if a certain structure fulfills this wishlist, we call it a vector space.

Definition 1 (group). Let G be an arbitrary set. $(G, +, 0)$ is called a group, if

1. there is an operation
$$+\colon G \times G \to G\,;$$
2. for all $x, y, z \in G$:
$$x + (y + z) = (x + y) + z\,;$$
3. there is an element $0 \in G$, such that for all $x \in G$:
$$0 + x = x + 0 = x\,.$$
0 is called "neutral element";
4. for every $x \in G$ exists a $(-x) \in G$, such that
$$x + (-x) = (-x) + x = 0$$
(existence of inverse elements).

Conditions 1–4 are called group axioms and will be explained now:
$+\colon G \times G \to G$ is a notion for "+ is an operation, which you give two elements of G and you obtain an element of G". An example would be: $g\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. g is the name of a function, we put two real numbers in and get another real number out. For example in the case of the +-operation: $g(7,3)\colon 7 + 3 = 10$.
The set has to contain a neutral element. For the +-operation (of the integers, or of the real numbers) it would be the 0, because if you add any number with 0, you always get this number out. If we defined the natural numbers without 0, they would fail this axiom, because they have no neutral element. However, the integers or the real numbers would fulfill this axiom, because they have 0 as a neutral element with respect to the +-operation. $-x$ would be the inverse element for x. The natural numbers would again fail this axiom, because they

contain no inverse element, but for example the integers would fulfill it, because they have an inverse element for each element. For example we take 1 and its inverse element (-1). Both are contained in the integers. Now, if we add 1 and (-1), we get 0.

If for a group also holds this fifth axiom, the commutativity, you call it a commutative group.

5. For all $x, y \in G$:

$$x + y = y + x$$

(commutativity of $+$).

The fifth axiom tells us, that for all $x, y \in G$ holds the commutativity in a $+$-operation. That means, that we get the same result, if we add x and y or y and x. So: $x + y = y + x$. An example with numbers would be: $4,2 + 8,79 = 8,79 + 4,2$.

But there are four further axioms, which should be fulfilled, to fix a vector space.

Definition 2 (vector spaces). Let G be a commutative group.

1. There is an operation – the scalar multiplication:

$$\cdot \colon \mathbb{R} \times G \to G\,.$$

2. There has to be a neutral element with respect to the scalar multiplication. So for all $x \in G$:

$$1 \cdot x = x\,.$$

3. For all $m, n \in \mathbb{R}$ and for all $x, y \in G$:

$$(m + n) \cdot x = m \cdot x + n \cdot x$$

and also

$$m \cdot (x + y) = m \cdot x + m \cdot y\,.$$

4. $\cdot$ is associative, such that for all $m, n \in \mathbb{R}$ and for all $x \in G$:

$$(m \cdot n) \cdot x = m \cdot (n \cdot x)\,.$$

$\cdot \colon \mathbb{R} \times G \to G$ is a notion for "$\cdot$ is an operation, which you give a real number and an element of the set G and you obtain an element of G". This scalar multiplication you already saw in vector algebra. For e.g. $2 \in \mathbb{R}$ and v a vector:

$$\cdot(2, v) \colon 2 \times v = 2 \times \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2v_1 \\ 2v_2 \end{pmatrix}.$$

1 is an element of the real numbers and in multiplication and scalar multiplication also the neutral element.

So here are four different operations involved: scalar multiplication, multiplication in $\mathbb{R}$, addition in $\mathbb{R}$ and vector addition.

If there is any set, which fulfills all axioms, it is a real vector space!

$(G, +, 0, \cdot)$ is called a real vector space. G is the set and consists of vectors, $+$ for the $+$-operation and has the "0-vector" as a neutral element, $\cdot$ for the scalar multiplication in the first axiom.

General remarks about groups

Bendix Labeit

We've already looked at the definition of a group. Now we want to further investigate what groups actually "are".

Simply spoken groups are sets of mathematical objects of which two can be connected such that you obtain another object. Of course groups were already used before the notion "group" was formally defined.

One of the very first applications of mathematics is the addition of numbers. If you further investigate this idea, you see that this is nothing more than having an "operation" (the addition) on a sets of mathematical objects (e.g. the integers). This is the basic idea behind the concept of groups. As people already knew a lot about these certain structures, the choice of the group-axioms is based on experience according to the integer numbers. (one example for a group is the set of the integers. It is the group which probably was used first).

To prove a set to be a group, you have to find out if the set connected with the operation fulfills all the group-axioms.

Below we there is a list of groups with each an explanation why it is a matter of a group.

Example 3 ($\mathbb{Z}, +$). We now consider the set $\mathbb{Z}$ together with the $+$-operation. As we know, $+$ is associative and commutative, 0 is a neutral element with respect to $+$ and for every element there is an inverse element. All axioms are fulfilled, therefore the set is a group.

Example 4 ($\mathbb{R}\backslash\{0\}, \times$). The set of the real numbers without the 0 together with the multiplication is also a group, because (just like in example 1) all group-axioms are fulfilled

Example 5. An "abstract" group:

Now we consider $(V, \%)$ to be a group. $V = \{\star, \heartsuit\}$. Let the operation $\%$ be defined by:

$$\begin{aligned} \%\colon V \times V &\to V \\ \star\%\star &:= \heartsuit \\ \star\%\heartsuit &:= \star \\ \heartsuit\%\star &:= \star \\ \heartsuit\%\heartsuit &:= \heartsuit \end{aligned}$$

Note that

1. there is an operation: $V \times V \to V$;

2. the operation is associative;

3. there is a neutral element: As we already know the neutral element can be recognized if we do the operation with an arbitrary element x and the neutral element, then we receive the element x we used. We have the operation:

$$\begin{aligned} \star\%\heartsuit &:= \star \\ \heartsuit\%\star &:= \star \\ \heartsuit\%\heartsuit &:= \heartsuit \end{aligned}$$

   As we can see, $\heartsuit$ fulfills this criterion. Therefore $\heartsuit$ must be the neutral element;

4. for every element there is an inverse element: We already know that if we do the operation with an element and its inverse element, then we receive the neutral element

$$\begin{aligned} \star\%\star &:= \heartsuit \\ \heartsuit\%\heartsuit &:= \heartsuit \end{aligned}$$

   As $\heartsuit$ is the neutral element, $\star$ must be the inverse element of $\star$ and $\heartsuit$ must be the inverse element of $\heartsuit$.

Example 3 shows, that the elements of groups don't have to be numbers. They can be just anything as long as the set together with the operation fulfills the group axioms. There are a lot of groups which consist of "abstract" elements.

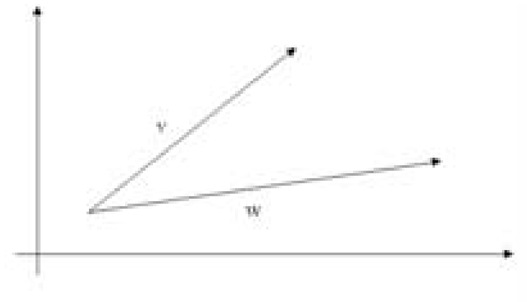
Linear independence

Lauretta Schwarz

Definition 3 (linear independence). Let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ be elements of a real vector space V. $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ are called linearly independent, if for all $a_1, \ldots, a_n \in \mathbb{R}$:

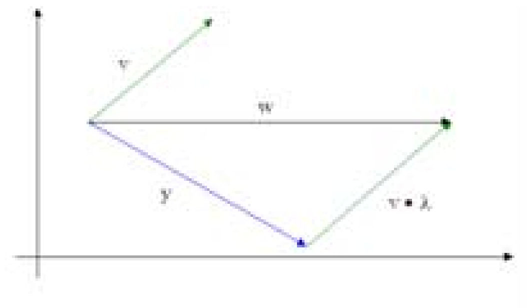
$$\begin{aligned} & a_1 \cdot \vec{v_1} + a_2 \cdot \vec{v_2} + \ldots + a_n \cdot \vec{v_n} = 0 \\ \Rightarrow\ & a_1 = a_2 = \ldots = a_n = 0\,. \end{aligned}$$

This means: If you have an arbitrary number of linearly independent vectors, you can scale them like you want and afterwards add them, but you will never get $\vec{0}$ as a result. You can only get the zero vector as a result of the addition, if all the scalars equal zero. For instance, two vectors in the two-dimensional space are linearly independent if they don't have the same direction.



These two vectors are linearly independent.

In the two-dimensional space three vectors are always linearly dependent.



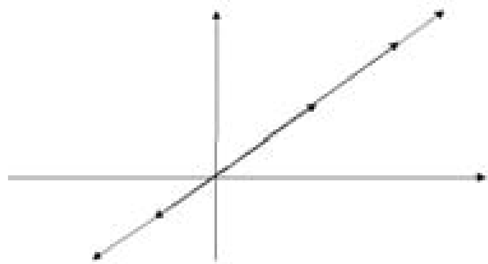
These three vectors are linearly independent because you can scale them, such that you get the zero vector as a result.

Definition 4 (linear span). Let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ be elements of V.

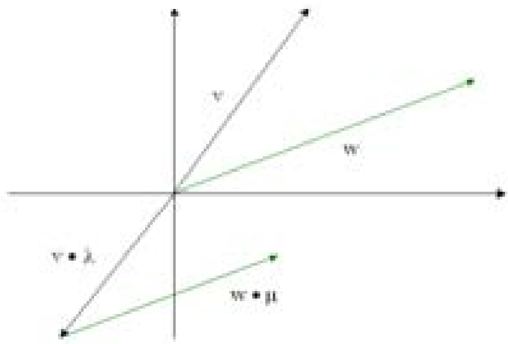
$$\langle \vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \rangle := \{a_1 \cdot \vec{v_1} + \ldots + a_n \cdot \vec{v_n} \mid a_1, \ldots, a_n \text{ are elements of } \mathbb{R}\}$$

is called the linear span of $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$.

This means: The linear span of an arbitrary collection of vectors is the set of all points that can be reached by scaling and adding these vectors. As an illustration: Two vectors in the two-dimensional space which have the same direction don't span the whole space, but a line.



Two vectors in the two-dimensional space which don't have the same direction span the whole space.



Dimensions

Christoph Sünderhauf

We'll need some other elementary definitions to be able to understand dimensions.

Let's define the basis B of a real vector space V:

Definition 5 (basis). A subset $B \subset V$ of the vector space is called basis of V, if B is linearly independent (all vectors in B are linearly independent) and if $\langle B \rangle = V$.

What does that mean? Some vectors of V are in the set B. Now we require their linear span to be V. So, if we scale all the vectors in B somehow and add them, then we get all vectors in V.

Wow! We can create V using B. But B has another special property too: B has to be "big" enough to span the whole vector space, but also "small" enough to be linearly independent.

So now we finally come to the definition of dimension in the mathematical sense.

Definition 6 (dimension). Let B be a basis of a vector space V. Then the dimension of $V = |B|$[3]

So we just used the previous definition of a basis. B is a basis of V, just like we defined before. $|B|$ is the minimal number of vectors needed to span V. Now we just define that this number is the dimension of V. Now we would have to proof that every basis of V has the same cardinality, but for convenience sake we won't state it here, because it's very complicated.

Let's use an example to make all this more clear. We'll use $\mathbb{R}^2$ to illustrate this. To find out the dimension of $\mathbb{R}^2$, we need a basis of $\mathbb{R}^2$. Let's consider the set consisting of the two vectors $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}3\\3\end{pmatrix}$.

A basis has to be linearly independent, but these two vectors are linearly dependent, since

$$-3 \cdot \begin{pmatrix}1\\1\end{pmatrix} + 1 \cdot \begin{pmatrix}3\\3\end{pmatrix} = 0\,.$$

Therefore they can't form a basis.

Let's try the vectors $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}5\\4\end{pmatrix}$. These two vectors are linearly independent, because $\forall\, \lambda, \mu \in \mathbb{R}$:

$$\begin{aligned} \lambda \cdot \begin{pmatrix}1\\2\end{pmatrix} + \mu \cdot \begin{pmatrix}5\\4\end{pmatrix} &= 0 \\ \Leftrightarrow 1\lambda + 5\mu &= 0 \\ \wedge \quad 2\lambda + 4\mu &= 0\,. \end{aligned}$$

[3] $|M|$ is called the *cardinality* of M. If M has a finite number of elements, $|M|$ is the number of elements in M.

Now we just have to solve this linear system of equations.[4] The result is $\lambda = \mu = 0$.

What is their linear span? We can write an arbitrary Vector of $\mathbb{R}^2$ as $\begin{pmatrix}x\\y\end{pmatrix}$, where $x, y \in \mathbb{R}$. So, if we want to express that their linear span is $\mathbb{R}^2$, we would have to solve the equation

$$\begin{pmatrix}x\\y\end{pmatrix} = \lambda \cdot \begin{pmatrix}1\\2\end{pmatrix} + \mu \cdot \begin{pmatrix}5\\4\end{pmatrix}$$

for any $x, \in \mathbb{R}$:

$$\begin{aligned} \begin{pmatrix}x\\y\end{pmatrix} &= \lambda \cdot \begin{pmatrix}1\\2\end{pmatrix} + \mu \cdot \begin{pmatrix}5\\4\end{pmatrix} \\ \Leftrightarrow \lambda + 3\mu &= x \\ \wedge \quad 2\lambda + 4\mu &= y \end{aligned}$$

If we solve this linear system of equations[5] we get $\lambda = \frac{y}{6} - \frac{x}{3}$ and $\mu = \frac{8x}{3} + \frac{5y}{6}$. Because λ and μ exist for all possible $x, y \in \mathbb{R}$, we know that we can form every $\begin{pmatrix}x\\y\end{pmatrix}$ as $\lambda \cdot \begin{pmatrix}1\\2\end{pmatrix} + \mu \cdot \begin{pmatrix}5\\4\end{pmatrix}$. Therefore we also know, that the linear span of $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}5\\4\end{pmatrix}$ is $\mathbb{R}^2$

So we found a basis of $\mathbb{R}^2$.

Of course there are other bases of $\mathbb{R}^2$. One of them, $\{e_1, e_2\}$, where $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$, is called *standard basis*. Ironically, there's nothing special about it, other than the small whole numbers as components.

The definition of the dimension of a vector space tells us that the dimension of $\mathbb{R}^2$ is the number of vectors in its basis. There are two vectors in its basis, so the dimension of $\mathbb{R}^2$ is 2.

We can look at an example from $\mathbb{R}^3$, too. Here $\begin{pmatrix}1\\2\\0\end{pmatrix}$ and $\begin{pmatrix}5\\4\\0\end{pmatrix}$ wouldn't form a valid

[4] This exercise is left to the reader.

[5] This exercise is also left to the reader.

bases. They are linearly independent[6], but their linear span isn't $\mathbb{R}^3$.

For Example $\begin{pmatrix}5\\9\\0\end{pmatrix}$, $\begin{pmatrix}9\\5\\0\end{pmatrix}$ and $\begin{pmatrix}1\\1\\7\end{pmatrix}$ form a basis of $\mathbb{R}^3$, so we know that the dimension of $\mathbb{R}^3$ is 3.

Norm and Scalar Product

Jan Forstbauer

Just imagine a length measuring which gives us the length of a vector v described by a real number. Such a machine does really exist, we call this map norm.

Definition 7 (norm). Let V be a real vector space. A map $\|\cdot\|: V \to \mathbb{R}$ is called a norm if it fulfills the following axioms:

1. $\forall \lambda \in \mathbb{R}\, \forall v \in V: \|\lambda \cdot v\| = |\lambda| \cdot \|v\|$
2. $\forall v \in V: \|v\| = 0 \Rightarrow v = 0$
3. $\forall v, w \in V: \|v\| + \|w\| \geq \|v + w\|$

From elementary geometry (Pythagorean theorem), we know how to calculate the length of vectors. So now I'll present you a special norm based on the Pythagorean theorem. It is called the Euclidean norm.

Definition 8 (Euclidean norm). The map defined as

- $\|v\| = \sqrt{x^2 + y^2}$ (twodimensional plane)
- $\|w\| = \sqrt{x^2 + y^2 + z^2}$ (threedimensional space)

is called the Euclidean norm.

[6] They're the same as in the example from $\mathbb{R}^2$, because their z is 0. So we already proved that they are linearly independent.

Instead of proofing that the Euclidean norm fulfills all these axioms – which is indeed the case – we'll now consider an application.

You might know the situation that you see a tree in the noon not standing straight up and you wonder how long the tree's shadow would be.

To solve that problem, we need another definition.

Definition 9 (scalar product). The scalar product of two vectors $v, w \in \mathbb{R}^n$ where $n = 2, 3$ is defined as:

$$v \cdot w = \|v\| \cdot \|w\| \cdot \cos \alpha$$

As result we get a real number representing the length of the projection of v onto w multiplied by the length of $\|w\|$.

If the scalar product is zero, there are two possibilities. The first is that one vector is orthogonal to the other. The second possibility is that one vector is the so called zero vector.

With help of this definition, we now are able to calculate the length of the tree's shadow. Both the tree and the ground can be represented as vectors v and w. The scalar product of v and w then yields the length of the shadow.

Sequences

Annika Konzelmann

Definition 10 (sequence). Let M be a set. A sequence in M is a map $a\colon \mathbb{N} \to M$. We usually denote a sequence by $(a_i)_{i\in\mathbb{N}}$.

This means that we have a machine in which we put a natural number i and receive an element a_i of the set M. A sequence can be illustrated by a list of infinitely many elements, of course elements may occur several times.

To make this clear we present the following examples:

Example 6. Let $M = \{\star, \heartsuit\}$.

The following table shows the position of each element.

position	element of M
1	$\star$
2	$\heartsuit$
3	$\star$
4	$\heartsuit$
...	...

Example 7. Let $M = \mathbb{R}^2$.

$a_i := \vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ for all $i \in \mathbb{N}$

Here you can see again the position of each element.

position	element of M
1	$\vec{v}$
2	$\vec{v}$
3	$\vec{v}$
4	$\vec{v}$
...	...

This is called a constant sequence.

Example 8. Let $M = \mathbb{N}$. The following sequence is called a Fibonacci Sequence:

$$a_1 = 1; \quad a_2 = 1;$$
$$a_i = a_{i-1} + a_{i-2} \text{ for } i \geq 3$$

This is called a recursive definition.

The position of each element is shown in the following table.

position	element of M
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55
...	...

After we worked with sequences we considered convergent sequences, which are needed for the definition of derivatives.

Definition 11 (accumulation point). Let $(a_i)_{i\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$. $x \in \mathbb{R}^n$ is called accumulation point if every disc with arbitrary radius around x covers infinitely many points of the sequence.

Definition 12 (convergent sequence). Let $(a_i)_{i\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$ $(n = 1, 2, 3)$. It is called convergent to the limit $x \in \mathbb{R}^n$, if

$\forall\, \varepsilon \in \mathbb{R}_{>0} \exists N \in \mathbb{N}\colon \forall i \geq N\colon ||a_i - x|| < \varepsilon.$

This formula says that for all possible radiuses $\varepsilon \in \mathbb{R}_{>0}$ there is a natural number such that for all list entries (elements of the sequence) the point a_i lies in the circle with radius ε around the point x.

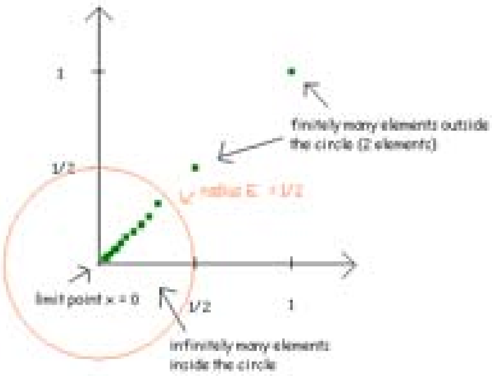
If the sequence is convergent and approaches x, then for all possible circles around x there are only finitely many elements of the sequence outside the circle and infinitely many elements inside the circle. The elements inside the circle accumulate to x, so x is the accumulation point.

Definition 13 (limit point). The only accumulation point of a convergent sequence is called limit point and we write:

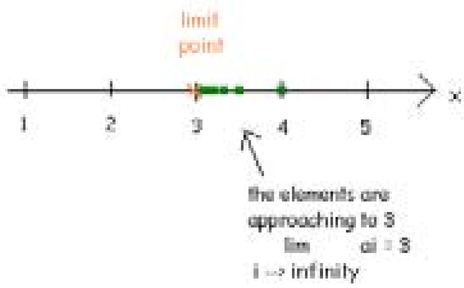
$$\lim_{i\to\infty} a_i = x.$$

Example 9.

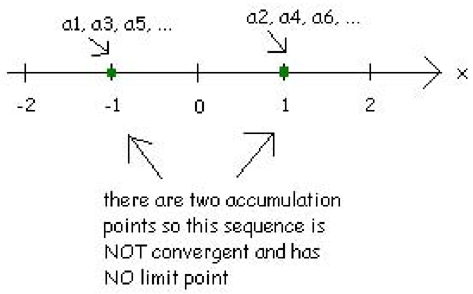
1. $M = \mathbb{R}^2$, $a_i = \begin{pmatrix} \frac{1}{i} \\ \frac{1}{i} \end{pmatrix}$



2. $M = \mathbb{R}$, $a_i = 3 + \frac{1}{i}$



3. $M = \mathbb{R}$, $a_i = (-1)^i$



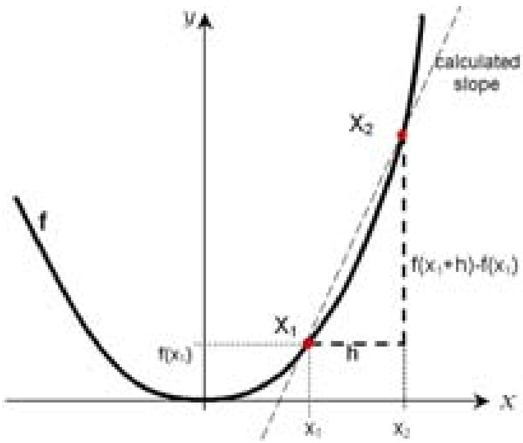
Differentiation

Mario Schulz

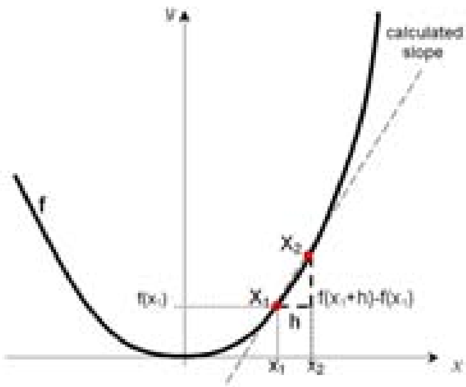
Motivation

At first we considered the case $\mathbb{R} \to \mathbb{R}$. Let f be a function $f\colon \mathbb{R} \to \mathbb{R}$ and $h \in \mathbb{R}$. Consider the points $X_1 = (x_1 \,|\, f(x_1))$ and $X_2 = (x_1 + h \,|\, f(x_1 + h))$.

The aim is to calculate the tangent slope at X_1. The average slope between X_1 and X_2 is calculated by $\frac{f(x_1+h)-f(x_1)}{h}$. This formula is deduced from the “slope-triangle”.



As you can see at the graph, the calculated slope would not be the tangent slope at X_1. We shorten then the distance between the points and now consider the point X_2 closer to X_1. As a result, the calculated slope approaches the tangent slope.

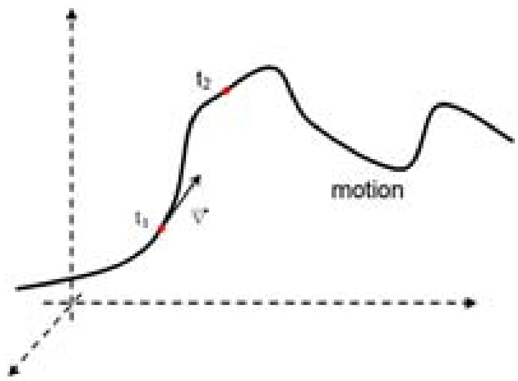


Therefore it would be necessary to put the second point just next to X_1 which means that h gets nearly 0. But h must not equal

0 because as you can see h is the nominator above. To solve this problem we introduced limits. The tangent slope m is then calculated by

$$m = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}.$$

Now consider a map $s : \mathbb{R} \to \mathbb{R}^n$ and $t_t, t_2 \in \mathbb{R}$.



The curve could be interpreted as the trajectory of some moving body in the n-dimensional space where every point of the curve gives the position of the body at some time t. The goal is to calculate the velocity v of the body at time t_1. An approximation for the velocity is calculated by $\frac{\Delta s}{\Delta t} = \frac{s(t_2)-s(t_1)}{t_2-t_1}$.
To get the exact velocity we shorten the "distance" between t_1 and t_2. If we let h be $t_2 - t_1$ it is possible to write the same like above:

$$v = \lim_{h \to 0} \frac{s(t_1 + h) - s(t_1)}{h}.$$

Geometrically the result would be the tangent vector at t_1. It is a vector because the velocity $v = \frac{\Delta s}{\Delta t}$, and Δs is the vector from $s(t_1)$ to $s(t_2)$ which is scaled by the time difference.

Definition (derivative). Let f be a function $f : \mathbb{R} \to \mathbb{R}^n$ where for $p \in \mathbb{R}$ the limit $\lim_{h \to 0} \frac{f(p+h)-f(p)}{h}$ exists. Then we call the function differentiable at p and $\frac{f(p+h)-f(p)}{h}$ is called the derivative of f at p. If this is possible for all $p \in \mathbb{R}$, the whole function is differentiable and its derivative, which is a new function, is called $f' : \mathbb{R} \to \mathbb{R}^n$. f' is called the first derivative, f'' (the derivative of the derivative) is called the second derivative of the function f.

Now consider the $s(t)$ diagram out of the "Hover-Car-Experiment" which gives us the distance s of the car after some time t. If the slope of this function at some point is calculated we obtain the change of s per change of t, which is the velocity. And in this case the slope is nothing else than the derivative at that point.

So the $v(t)$-diagram is just the first derivative of the $s(t)$-diagram. Further it is possible to take again the derivative: The result is the second derivative, which is the change of v per change of t, which equals to the acceleration.

So we can write:

$$s = s(t), \quad s' = v(t), \quad s'' = a(t),$$

and use this in physics.

Differentiation rules

David Kranzhöfer

Taking the derivative of a function $f(x)$ at the point x is called differentiation. The derivation of $f(x)$ at x involves calculations with limits. As calculating with limits is very difficult, we now give three simple differentiation rules summarized in two propositions. These rules make calculating derivatives more easy without using limits.

Proposition 1 (linearity of differentiation).

(i) $(a \cdot f(x))' = a \cdot f'(x) \quad a \in \mathbb{R}$

(ii) $(f + g)'(x) = f'(x) + g'(x)$

Rule (i) helps us in the case we have a formula in the form $a \cdot f(x)$. Then, we can neglect the factor a in front of the function for the derivation and only take the derivative of the function $f(x)$ which is called $f'(x)$. Afterwards we have to multiply the derivative by the factor a.

In the case we want to know the derivative of the sum of two functions $f(x)$ and $g(x)$, we may use the rule (ii), which is called sum rule. This rule tells us that for each arbitrary function $h(x) = (f + g)(x)$, we obtain the derivative $h'(x)$ by splitting up the function into two new functions $f(x)$ and $g(x)$ and sum up the derivatives of each of these two functions.

Example 10. We proved that the derivative of $f(x) = x^2$ is $2x$. (You can find the proof for functions of the form $f(x) = x^n$ at the end of this chapter.) In the following example, we apply rule (i).

$$\begin{aligned} &3 \cdot f(x) = 3x \\ &(3 \cdot f)'(x) = 3 \cdot f'(x) = 3 \cdot 2x = 6x \end{aligned}$$

Example 11. The next example shows us an application of rule (ii).

$$\begin{aligned} h(x) &= 3x^2 = 2x^2 + x^2 = f(x) + g(x) \\ \Rightarrow \quad h'(x) &= (f+g)'(x) \\ &= f'(x) + g'(x) \\ &= 4x + 2x \\ &= 6x \end{aligned}$$

Proposition 2 (product rule). For all differentiable functions $f(x)$, $g(x)$ from $\mathbb{R}$ to the $\mathbb{R}$ holds:

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Beweis. The proof of the product rule mainly consists of difficult calculations:

$$\begin{aligned} (f \cdot g)'(x) &= \lim_{h \to 0} \frac{(f \cdot g)(x+h) - (f \cdot g)(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h} \\ &= \lim_{h \to 0} \Big[\frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h} \\ &\quad + \frac{f(x+h) \cdot g(x)}{h} - \frac{f(x+h) \cdot g(x)}{h} \Big] \\ &= \lim_{h \to 0} \Big[\frac{f(x+h) \cdot g(x+h)}{h} - \frac{f(x+h) \cdot g(x)}{h} \\ &\quad + \frac{f(x+h) \cdot g(x)}{h} - \frac{f(x) \cdot g(x)}{h} \Big] \\ &= \lim_{h \to 0} f(x+h) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &\quad + g(x) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} f(x+h) \cdot g'(x) + g(x) \cdot f'(x) \\ &= f(x) \cdot g'(x) + g(x) \cdot f'(x) \end{aligned}$$

□

The product rule is used in case we have a function $h(x)$ which is the product of two functions $f(x)$ and $g(x)$. Then, we can split it up into the functions $f(x)$ and $g(x)$ which we already know the derivatives of. (In the following example, we know the derivatives of $f(x)$ and $g(x)$ by the help of the proof at the end of this chapter.) With the help of this knowledge, we can use the formula above to calculate the derivative of $h(x)$. Example:

$$\begin{aligned} h(x) &= 2x^3 = 2x \cdot x^2 = f(x) \cdot g(x) \\ \Rightarrow \quad (f \cdot g)'(x) &= f'(x) \cdot g(x) + f(x) \cdot g'(x) \\ &= 2 \cdot x^2 + 2x \cdot 2x \\ &= 2x^2 + 4x^2 \\ &= 6x^2 \end{aligned}$$

Additionally, we want to have a look at the generalized case when we have a function of the form $f(x) = x^n$ as we had in all examples above. If we know the derivative of this generalized case $f(x) = x^n$, it'll be very simple to calculate the derivative of a certain function of this form.

Proposition 3. Let $f(x) = x^n$ be a differentiable function. Then the derivative is $f'(x) = nx^{n-1}$

Beweis. We do this proof by mathematical induction.

Start of induction

We have to proof that the assumption holds for $n = 1$, i.e. $f(x) = x$. The derivative has to be $f'(x) = 1$ which indeed is the case, as was shown earlier.

Induction step

Now comes the more difficult part of the induction. We assume that the assertion holds for n. Then we have to prove that it holds for $n + 1$.

$$f(x) = x^{n+1}$$

We split this function into two functions $f_1(x) = x^n$ and $f_2(x) = x$ such that

$$f_1(x) \cdot f_2(x) = f(x).$$

As we already know the derivatives of these two functions (of x^n because we assumed it at the beginning of the induction step, of x because we proved in the start of induction), we can use the product rule $f'(x) = f_1(x) \cdot f_2'(x) + f_1'(x) \cdot f_2(x)$.

$$\begin{aligned} f'(x) &= x^n \cdot 1 + nx^{n-1} \cdot x \\ &= x^n + nx^n \\ &= (n+1)x^n \end{aligned}$$

By mathematical induction follows that the derivative of the function $f(x) = x^n$ is $f'(x) = nx^{n-1}$. □

Newtonian Axioms

Anastasia Dietrich

One of the most important sectors of classical physics is mechanics. And there wouldn't be something like this if there weren't the Newtonian Axioms.

First Newtonian Axiom

"A body tends to stay at rest and it tends to stay in uniform motion unless there is a force acting on it."

If you throw a ball, according to the First Newtonian Axiom this ball should continue to move in the same direction with a constant velocity. Obviously this isn't the usual behavior of balls on earth, which does not mean that this axiom is wrong. The axiom is still valid because on earth there *is* a force acting on it, as we always have gravitation and friction.

Second Newtonian Axiom

Imagine you are climbing in the rocks. You slip off and fall down into your safety rope, which does not hold you. It did indeed as you tried before if it would hold you. But why not now? There must have been different momentums.

But what is *momentum*?

1. Think of a car with a big mass and a car with a low mass. Both of them are driving with the same velocity before they crash into a wall. The question is which one gets damaged more. Of course the answer is the one with the bigger mass. And that would mean that the car with the big mass had more momentum than the other car, although they were driving with the same velocity. And this tells us that momentum is also dependent on mass.

2. Think of a fast and a slow car with the same mass, which are traveling on the freeway. Now they crash into a wall. Which one is damaged more? Of course the one with the higher velocity. That means that the fast car had more momentum. Because there was no difference between the masses, velocity must be also one factor of momentum.

What is *force*?

Imagine that you are a sprinter. You want to run 20 km/h over a short distance. Now imagine that you have two possibilities. The first one is to accelerate your body in a short period of time to this momentum which you get, if you run 20 km/h. The second one is to accelerate you body slower to this velocity and momentum. For which one would you need more force? Of course for the faster acceleration. It is the same with a sport car which runs up to 100 km/h in $7s$ and a Golf which needs $13s$. The sport car's engine has more power and is therefore able to apply more force during accelerating. Force is the change of the momentum divided by the time-difference.

$F = m \cdot a$, because $p = m \cdot v$ and $v' = a$ (v' is the derivative of v)

Gun-Experiment:

Two balls with the same mass are put in something like a gun, so that after releasing this gun one ball is shot on a horizontal path to the right and the other one just falls directly down to the ground.

We observe by listening:

Both balls hit the ground simultaneously. We can see that the motions in two orthogonal directions does not disturb each other. It is called the "superposition principle".

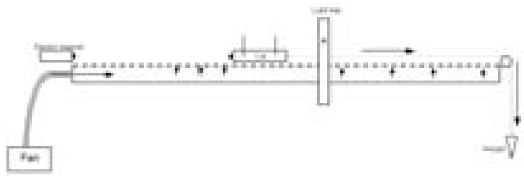
Hover-Car-Experiment

Jonathan Schaible

A nice experiment for discovering physical laws and mathematical interrelations is the Hover-Car-Experiment.

Description

The main element of this experiment is a rail with small holes out of which a fan blows air. Therefore, a small car can move on the rail with very little friction. It is



Diagrammatic picture of the experiment.

accelerated by a mass connected to the car by a rope. Now the time the car needs to move a certain distance can be measured by setting a light trap at this point. When the light beam is interrupted by a stick fixed at the car, an electronic time clock displays the time since it has been started.

For more precise measurements, we didn't try to start the car and the clock simultaneously ourselves. Instead, we used an electric magnet holding the car until the clock starts and cuts the magnet's current entry. What still could spoil our measurements the magnet's core is still a little magnetized if it is turned off which can decelerate the car or that the time difference between the two sticks crossing the light trap is too short for the clock to notice it.

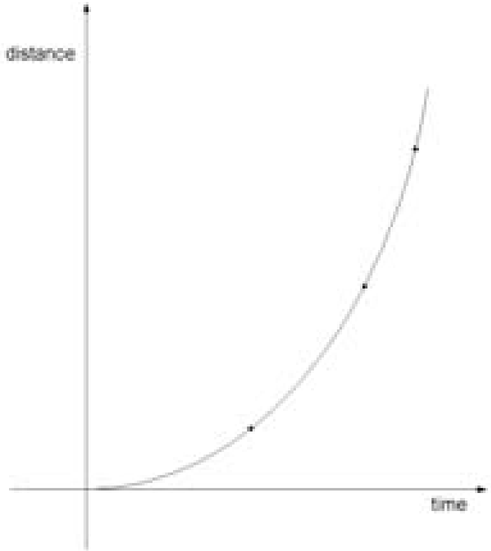
Trying out different masses to pull the car, different masses which can be packed on the car and different distances to the light trap in order to collect many values helps remark some interrelations.

In the following example measurements, *time 1* is the time between the first and the second stick of the car crossing the light trap and *time 2* is the time between the start and the first stick crossing the light trap.

Distance [m]	time 1 [s]	time 2 [s]
0,25	0,108	1,065
0,75	0,066	1,816
1,00	0,058	2,098

Mass (car) [g]	time 1 [s]	time 2 [s]
0	0,078	1,360
2	0,098	1,805
3	0,111	2,076

Mass (pulling) [g]	time 1 [s]	time 2 [s]
10	0,080	1,480
20	0,058	1,066
30	0,048	0,889

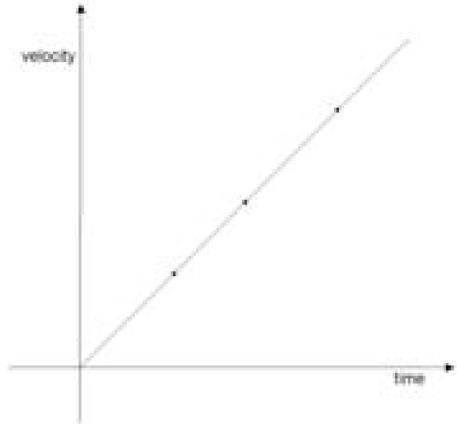


The distance depending on the time.

The points in this diagram all lie on a curve which resembles a parabola, so you will get different slopes at each point of time. The slopes of the *secants* (the lines connecting two points on the curve) of this presumed function are not the same as the slopes of the *tangents*, which means concretely: By drawing those secants, you always get the average speed for a short time and not the current speed. *Calculating* the slope instead of showing it geometrically is done by dividing the y-displacement by the x-displacement, in this case it would be $\frac{\Delta s}{\Delta t}$.

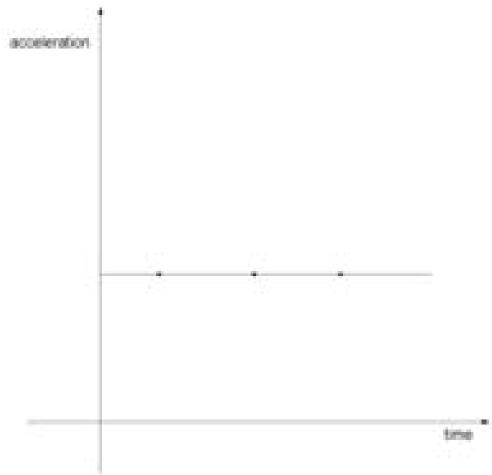
These values of the slopes just obtained can now be put into another diagram which shows the velocity depending on the time. In order to do that, you need several velocities at different points of time. Connecting those points in a diagram results in a straight line going through the origin. Thus, it is a linear function.

Again, you calculate the slope which is the same at each point of time in this very case. So the third step would be to draw a diagram showing the acceleration depending



The velocity depending on the time.

on the time. Some values are needed, too, and if you connect the points in a diagram, you will get a straight line parallel to the x-axis, a constant function. This is the case because the slopes in the second diagram are always the same.



The acceleration depending on the time.

But you can constitute this physically, too, because the second Newtonian axiom says: The rate of change of momentum of a body is equal to the resultant force acting on the body and is in the same direction.

The acceleration a must be constant as well: For the momentum p holds

$$p = m \cdot v.$$

And because the second Newtonian axiom just says that the force F is equal to the rate of change of momentum, you can write:

$$F(t) = p'(t).$$

(The force equals to the derivative of the momentum.) By using the product rule, we can calculate:

$$\begin{aligned} F(t) &= (m(t) \cdot v(t))' \\ &= m'(t) \cdot v(t) + m(t) \cdot v'(t) \\ &= m'(t) \cdot v(t) + m(t) \cdot a(t). \end{aligned}$$

The mass is constant in the present experiment, therefore its derivative equals to 0.

$$\begin{aligned} &\Rightarrow m'(t) \cdot v(t) = 0 \\ &\Rightarrow F(t) = m(t) \cdot a(t) \end{aligned}$$

Or, more easily:

$$F = m \cdot a.$$

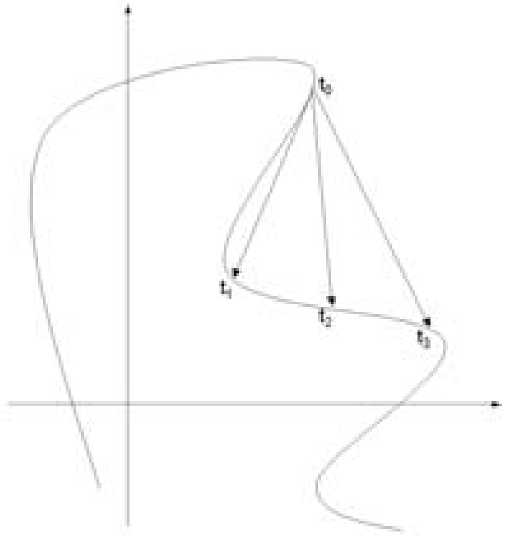
If F is the force (here: the gravitation force acting on the pulling mass), m is the accelerated mass of the car and a is the acceleration, then you can express this law in the formula $F = m \cdot a$. And as F and m are constant in this experiment, a is constant as well.

Differentiating

Taking the slope of a function and expressing it in a new function as in the previous example is called differentiating. The function which assigns the slope to a point of time is called derivative. The derivative is obtained by differentiating at each point of the function. So to get from the first to the second function in the example with the hover car experiment, you take the first derivative and from the second one to the third one, you take the second derivative of the first function.

Of course all this is still very imprecise. To calculate the exact derivatives, you need differential calculus.

Another example where differential calculus is needed is an ant moving on a two-dimensional plane.



Trajectory of an ant moving on a plane.

To describe the trajectory of the ant, you need a function mapping from the time to the space:

$$s\colon \mathbb{R} \to \mathbb{R}^2.$$

The velocity $\vec{v}$ at time t_0 can be described like that:

$$v = \frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{s(t_0 + h)}{h}, \quad t_1 - t_0 = h.$$

If h gets smaller and smaller, the calculation gets more precise: The hypotenuse gets more and more tangent to the curve at the point t_0. But h must never equal to 0 as it isn't possible to divide by 0. In this case you need again the differentiation.